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# Squeezing and broadening effects in mechanical oscillators

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**Abstract**
The general theory of harmonic oscillators with time-dependent frequency and 'effective mass' is related in the present work to the use of the adiabatic approximation. A slow 'pumping' of energy into the mechanical oscillator leads to a slow increase of the amplitude and frequency with squeezing effects by which the noise in one quadrature is increasing and in the other quadrature is decreasing. An order of magnitude for the squeezing effect is calculated. It is suggested to use this theory for mechanical oscillators in traps and for resonant detectors. The other extreme case of 'impulse' interaction in which the time of interaction is very small relative to the time period of the harmonic oscillator is analyzed. In the 'impact' approximation the time development is without any squeezing, but broadening effects might be important under special conditions. The 'impact' approximation might be related to the detection of gravitational waves by Michelson interferometers.

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## 1. Introduction

The problem of a harmonic oscillator with time-dependent mass has been related to a quantum-damped oscillator [1–7]. In these studies the mass parameter is given as a general function of time, and the equations of motion include a damping mechanism with a driving force [7] or without it [6]. As these systems are non-conservative, quantum theories of non-conservative systems have been developed.

The study of problems involving harmonic oscillators with time-dependent masses [8, 9], or with time-dependent frequencies [10–19], or both simultaneously [20–28] has attracted much interest with the various publications. Of course such systems are not closed in the sense that some external influence, which not need be specified, may change the harmonic oscillator parameters, i.e., alter its frequency, amplitude, etc. By using the generalized quantum-mechanical invariant, first introduced by Lewis [13, 14] and Lewis and Reisenfeld [11], the exact quantum states for such oscillators can be found for various special cases. It is shown that squeezing phenomena (i.e., quantum fluctuations in one quadrature which are smaller relative to those of the other quadrature) will inevitably be generated in a harmonic oscillator with time-dependent frequency and mass. Squeezing phenomena have been found in the analysis of harmonic oscillators with variable mass [8, 9] or variable frequency [10–19], or with both [20–28]. Variable frequencies in electromagnetic oscillators are well known in the field of optics [29] and have been referred as 'chirping'. In the present paper we are interested, however, in squeezing effects in mechanical oscillators. Squeezing phenomena of mechanical oscillators have been analyzed and observed in the quantum dynamics of single trapped ions [30].

In the present work we would like to study, especially, squeezing phenomena which can occur in *mechanical oscillators* which are produced by slow pumping of energy in the mechanical oscillator and which can be analyzed by the use of the adiabatic approximation. While quantum harmonic oscillators with time-dependent mass ('effective mass') have been extensively studied [1–28] the use of the quantization by the adiabatic approximation has not been exploited. The use of the adiabatic approximation simplifies the analysis very much and gives simple quantitative results for the magnitude of the squeezing effects.

The use of the adiabatic approximation is compared with the other extreme case in which the 'impulse' driving force leads to a shift of the wave packet in the momentum $k$ distribution without any squeezing (but broadening effects in the $x$ representation might be important, under special conditions). The use of such approximation for gravitational waves detection in Michelson interferometers is discussed.

The present paper is arranged as follows: in section 2 we present a general theory of a harmonic oscillator with time-dependent frequency and effective mass. In section 3, we analyze the squeezing effects produced in mechanical oscillators due to time-dependent frequency ('chirping') and/or time-dependent effective mass (i.e., slow change in the harmonic oscillator amplitude due to damping or energy pumping). In section 4, we analyze the squeezing effects obtained by slow pumping of energy into the harmonic oscillator using the adiabatic approximation. The adiabatic approximation represents a slow variation of the harmonic oscillator parameters so that the harmonic oscillator remains as an eigenstate of the Hamiltonian (1). In section 5, we describe the other extreme case where the external perturbation can be considered as an 'impulse' function and the time of the external perturbation is small relative to the time period of the harmonic oscillator. The use of the 'impulse' approximation for gravitational waves detection in Michelson interferometers is discussed. In section 6, the present results are summarized and possible implications of the present theories to mechanical oscillators and resonant detectors are discussed.

## 2. The evolution operator of a harmonic oscillator with time-dependent frequency and effective mass

We consider the following Hamiltonian of a harmonic oscillator:

$$\hat{H}(t) = \frac{\hat{p}^2}{2M_{\text{eff}}(t)} + \frac{1}{2}M_{\text{eff}}(t)\omega(t)^2\hat{q}^2, \tag{1}$$

where $M_{\text{eff}}(t)$ and $\omega(t)$ are the effective mass and frequency, respectively, and they are time dependent. This Hamiltonian has been investigated in many articles [6–28] and we use here the analysis given in [20–22]. We rewrite the Hamiltonian (1) as

$$\hat{H}(t) = a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_-, \tag{2}$$

where

$$\hat{J}_+ = \frac{\mathrm{i}}{2\hbar}\hat{q}^2, \qquad \hat{J}_- = \frac{\mathrm{i}}{2\hbar}\hat{p}^2, \qquad \hat{J}_0 = \frac{\mathrm{i}}{4\hbar}(\hat{p}\hat{q} + \hat{q}\hat{p}) \tag{3}$$

and

$$a_1(t) = -\mathrm{i}\hbar M_{\mathrm{eff}}(t)\omega(t)^2, \qquad a_2(t) = 0, \qquad a_3(t) = \frac{-\mathrm{i}\hbar}{M_{\mathrm{eff}}(t)}. \tag{4}$$

Here $\hat{J}_+$, $\hat{J}_0$ and $\hat{J}_-$ form the SU(1,1) Lie algebra satisfying the commutation relations (CR):

$$[\hat{J}_+, \hat{J}_-] = -2\hat{J}_0, \qquad [\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm. \tag{5}$$

The Schrödinger equation corresponding to this Hamiltonian is

$$\hat{H}(t)|\phi(t)\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\phi(t)\rangle. \tag{6}$$

The evolution operator $\hat{U}(t,0)$ is given by

$$|\phi(t)\rangle = \hat{U}(t,0)|\phi(0)\rangle, \tag{7}$$

where $|\phi(0)\rangle$ is the wavefunction at time $t = 0$. Insertion of (7) into (6) gives the evolution equation

$$\hat{H}(t)\hat{U}(t,0) = \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{U}(t,0), \qquad \hat{U}(0,0) = 1. \tag{8}$$

Since $\hat{J}_+$, $\hat{J}_0$ and $\hat{J}_-$ form a closed $SU(1,1)$ Lie algebra, the evolution operator can be expressed in the following form:

$$\hat{U}(t,0) = \exp(c_1(t)\hat{J}_+)\exp(c_2(t)\hat{J}_0)\exp(c_3(t)\hat{J}_-), \tag{9}$$

where differential equations for $c_i(t)$ $(i = 1, 2, 3)$ are obtained by direct differentiation of this operator with respect to time and by substituting the result into (8). By certain reordering of the operators on the right-hand side of (8) and comparing them with the left-hand side of this equation one gets three differential equations for $c_1(t), c_2(t), c_3(t)$, where their solutions can be presented as [20]:

$$\begin{aligned} c_1(t) &= M_{\mathrm{eff}}(t)\frac{1}{u(t)}\frac{\partial}{\partial t}u(t), \qquad c_2(t) = -2\ln\left|\frac{u(t)}{u(0)}\right|, \\ c_3(t) &= -u^2(0)\int_0^t \frac{\mathrm{d}u}{M_{\mathrm{eff}}(t')u^2(t')}. \end{aligned} \tag{10}$$

Here $u(t)$ satisfies the following auxiliary differential equation:

$$\ddot{u} + \gamma_0(t)\dot{u} + \omega(t)^2 u = 0 \tag{11}$$

and

$$\gamma_0(t) = \frac{\partial}{\partial t}[\ln(M_{\mathrm{eff}}(t)] = \frac{1}{M_{\mathrm{eff}}(t)}\frac{\partial}{\partial t}M_{\mathrm{eff}}(t). \tag{12}$$

The auxiliary equation (11) for the present analysis is quite complicated and does not have a general solution. We refer to the literature [11–14] for the solution of this equation for special cases. For our purpose we shall simplify, however, our analysis in section 4 by the use of the adiabatic approximation.

Using cgs units we should note that in equations (10), $c_1(t)$ has the dimension g/s, $c_3(t)$ has the dimension s/g, while $c_2(t)$ is dimensionless. Vice versa, in the definitions (3) $\hat{J}_+$ has the dimension s/g, $\hat{J}_-$ has the dimension g/s while $\hat{J}_0$ is dimensionless.

## 3. Coherence and squeezing properties of the wavefunction of a harmonic oscillator with time-dependent frequency and effective mass

The following analysis for describing the coherence and squeezing properties of a harmonic oscillator with time-dependent frequency and effective mass is based on the use of the $SU(1,1)$ algebra [20]. Similar results are obtained by the use of the SU(2) algebra [8]. The fact that the time dependence of the Hamiltonian (1) by the use of the SU(1,1) algebra [20, 21] gives the same results as those obtained by the use of the SU(2) algebra seems to be quite surprising. However, one should note that the decomposition of $\hat{H}(t)$has been made in [2–4] only into the operators $\hat{J}_+$ and $\hat{J}_-$, while $\hat{J}_0$ does not appear in this decomposition (as $a_2(t) = 0$). One can therefore replace the operators $\hat{J}_+$ and $\hat{J}_-$ by the operators [8]:

$$\hat{J}_+ = \frac{1}{2\hbar}\hat{q}^2, \qquad \hat{J}_- = \frac{1}{2\hbar}\hat{p}^2, \qquad \hat{J}_0 = \frac{\mathrm{i}}{4\hbar}(\hat{p}\hat{q} + \hat{q}\hat{p}), \tag{3a}$$

which satisfy the SU(2) algebra, as was done in [8], and the same Hamiltonian $\hat{H}(t)$ is obtained by assuming [8]

$$a_1(t) = \hbar M_{\mathrm{eff}}(t)\omega(t)^2, \qquad a_2(t) = 0, \qquad a_3(t) = \frac{\hbar}{M_{\mathrm{eff}}(t)}. \tag{4a}$$

Since both decompositions (those of [8, 20]) describe the same Hermitian Hamiltonian they give equivalent physical results.

Using the evolution operator (9) in the SU(1,1) representation we study the time development of a harmonic oscillator with time-dependent frequency and effective mass. We use the definitions:

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega M}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = -\mathrm{i}\sqrt{\frac{\hbar\omega M}{2}}(\hat{a} - \hat{a}^\dagger), \tag{13}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and the creation operators, respectively. Suppose we start with a coherent state at time $t = 0$:

$$|\phi(0)\rangle = |\alpha\rangle, \tag{14}$$

where $|\alpha\rangle$ is the eigenstate of the annihilation operator and $\omega = \omega_0$ is the frequency at time $t = 0$. $M$ is the ordinary mass which is equivalent to the effective mass at time $t = 0$, i.e.,

$$M_{\mathrm{eff}}(0) = M. \tag{15}$$

We can define a new operator $\hat{A}$ as

$$\hat{A} = \hat{U}(t,0)\hat{a}U^\dagger(t,0). \tag{16}$$

It is easy to see that the wavefunction at time $t$ is a coherent state with respect to the new operator

$$\hat{A}|\phi(t)\rangle = \alpha|\phi(t)\rangle. \tag{17}$$

Using (9) and (16) it can be shown [8, 20, 21] that the original operator $\hat{a}$ is related to the new operator $\hat{A}$ by Bogoliubov transformation

$$\hat{A} = \eta_1\hat{a} - \eta_2\hat{a}^\dagger, \tag{18}$$

with

$$|\eta_1|^2 - |\eta_2|^2 = 1. \tag{19}$$

$\eta_1$ and $\eta_2$ have been calculated by the use of the SU(1,1) algebra [21] obtaining

$$\eta_1 = \frac{1}{2}\exp\left(-\frac{c_2}{2}\right)\left[1 - c_1c_3 + \exp(c_2) - \frac{\mathrm{i}c_1}{M\omega_0} - \mathrm{i}M\omega_0c_3\right], \tag{20}$$

$$\eta_2 = \frac{1}{2}\exp\left(-\frac{c_2}{2}\right)\left[1 + c_1c_3 - \exp(c_2) + \frac{\mathrm{i}c_1}{M\omega_0} - \mathrm{i}M\omega_0c_3\right]. \tag{21}$$

It is easy to verify that (19) is satisfied. One should take into account that although the expressions for $c_1(t), c_2(t), c_3(t)$ in [8] and [20, 22] are different as different algebras were used, the final results for $\eta_1$ and $\eta_2$ are the same in both references. In the present analysis we find it more convenient to follow the SU(1,1) representation [20–22].

The critical point in estimating the magnitude of the squeezing phenomena is the explicit calculation of the parameters $c_1(t)$, $c_2(t)$ and $c_3(t)$. As shown in the previous section this can be done only by solving the auxiliary differential equation (11) which has no general solution but can be solved either for specific cases or by assuming the adiabatic approximation where its use is developed in the following section.

## 4. Squeezing effects in mechanical oscillators under the adiabatic approximation

Squeezing effects can be produced in a mechanical harmonic oscillator with slowly varying amplitude and frequency. Such effects can be produced by slow pumping of energy into the harmonic oscillator. We describe first the effect of increasing the harmonic oscillator amplitude by classical equations and then describe the quantization of these equations which are related to the harmonic oscillator with time-dependent effective mass.

The displacement of the mechanical oscillator can be described as

$$q(t) = f(t)A(0)\cos(\omega(t)t + \phi_0), \qquad f(0) = 1, \quad \omega(0) = \omega_0, \tag{22}$$

where $f(t)A(0)$ represents a slowly growing amplitude with time with initial amplitude $A(0)$. In the adiabatic approximation the change of $f(t)$ and $\omega(t)$during the time period of oscillation is negligible but its effect is accumulated over many time periods.

The time derivative of $q(t)$, within the adiabatic approximation, can be given as

$$\dot{q}(t) = -f(t)A(0)\omega(t)\sin(\omega(t)t + \phi_0), \tag{23}$$

where we have neglected here the time derivatives of $f(t)$ and $\omega(t)$, but have taken into account their slow change with time. We introduce the definition

$$M_{\text{eff}}(t) = \frac{M}{f(t)}, \tag{24}$$

where $M$ is the ordinary mass and $M_{\text{eff}}(t)$ is defined as the effective mass. Then the equation of motion (23) becomes

$$\dot{q} = \frac{p}{M_{\text{eff}}}, \qquad p = -\frac{1}{2}MA(0)\omega(t)\sin(\omega(t)t + \phi_0), \tag{25}$$

where $p$ is the linear momentum. The equation of motion for $p$, under the adiabatic approximation, becomes

$$\dot{p} = -\tfrac{1}{2}MA(0)\omega(t)^2\cos(\omega(t)t + \phi_0) = -M_{\text{eff}}(t)\omega(t)^2 q. \tag{26}$$

The quantization of the present one-dimensional harmonic oscillator is given by the Hamiltonian operator

$$\hat{H}(t) = \frac{\hat{p}^2}{2M_{\text{eff}}(t)} + \frac{1}{2}M_{\text{eff}}(t)\omega(t)^2\hat{q}^2, \tag{27}$$

which is equivalent to the Hamiltonian (1). One should, however, take into account that in our analysis the effective mass $M_{\text{eff}}(t)$ is smaller than $M$ as it simulates energy pumping (increasing amplitude) while in most treatments of the Hamiltonian (1) it simulates energy damping (decreasing amplitude).

The derivation of the Hamiltonian (27), by using the adiabatic approximation, has a simple physical explanation: the equations of motion for $\dot{p}$ and $\dot{q}$ are similar to those of ordinary harmonic oscillators but in (27) $M$ and $\omega$ have been changed into $M_{\mathrm{eff}}(t)$ and $\omega(t)$, respectively, and in (22) $A(0)$ has been changed into $f(t)A(0)$ ($f(t) = M/M_{\mathrm{eff}}(t)$). So, in the adiabatic approximation the oscillator follows the harmonic oscillator equations with the slowly varying parameters $M_{\mathrm{eff}}(t)$ and $\omega(t)$.

Using the quantized Hamiltonian (27) the variables $\hat{q}$ and $\hat{p}$ have become operators satisfying the commutation relations

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar, \tag{28}$$

and the canonical equations of motion are

$$\dot{\hat{q}} = \frac{1}{\mathrm{i}\hbar}[\hat{q}, \hat{H}] = \frac{p}{M_{\mathrm{eff}}}, \qquad \dot{\hat{p}} = \frac{1}{\mathrm{i}\hbar}[\hat{p}, \hat{H}] = -M_{\mathrm{eff}}(t)\omega(t)^2 q. \tag{29}$$

The quantized equations of motion (29) correspond to the classical equations of motion (25) and (26). While $\hat{q}$ and $\hat{p}$ are considered as operators, $M_{\mathrm{eff}}(t)$ and $\omega(t)$ are taken into account as classical variables.

Since the quantization of the present system is similar to the general analyses of time-dependent harmonic oscillators [1–28] we expect that there will be quantum squeezed noise effects in the present system which will be analogous to those analyzed previously [8–28] in other systems. We will estimate the magnitude of squeezing effects by the use of the adiabatic approximation.

For calculating the time-dependent parameters $c_1(t), c_2(t)$ and $c_3(t)$ of (10) we need first to find the solution for the auxiliary equation given by (11) and (12) which as we show simulates the classical second-order differential equation for $q$. Using (29) we get

$$\ddot{q} = \frac{\dot{p}}{M_{\mathrm{eff}}} - p\frac{1}{M_{\mathrm{eff}}^2}\frac{\mathrm{d}M_{\mathrm{eff}}}{\mathrm{d}t}, \qquad M_{\mathrm{eff}} = \frac{M}{f(t)}. \tag{30}$$

Substituting into (30), $\dot{p}$ from (26), $p$ from (25) and $\gamma_0$ from (12) then we get

$$\ddot{q} + \gamma_0\dot{q} + \omega(t)^2 q = 0 \tag{31}$$

so that the differential equation for the classical $q$ variable, within the adiabatic approximation, is equivalent to the differential equation for $u$.

In performing the first-order time derivative of $q$, or correspondingly $u$, we have neglected the time derivative of $f(t)$. In performing the second-order time derivative of $q$, or correspondingly $u$, we have taken into account the first-order time derivative of $f(t)$ multiplied by $\omega(t)$. This term is considered as a correction term proportional to $\gamma_0$ and $\dot{q}$. This approximation is obtained within the adiabatic approximation and is consistent with the above quantization procedure.

We use the relations

$$u(t) = u_0 f(t)\cos(\omega(t) + \phi_0), \qquad f(t) = \frac{M}{M_{\mathrm{eff}}}, \tag{32}$$

where $u(t)$ is the solution of (11) within the adiabatic approximation and where $f(0) = 1$. Substituting (32) into (10) and averaging over the oscillating trigonometric function of $c_2(t)$ we get in the SU(1,1) representation

$$c_1(t) = M_{\mathrm{eff}}(t)\frac{\partial}{\partial t}\{\ln[f(t)\cos(\omega(t)t + \phi_0)]\}, \tag{33}$$

$$c_2(t) = -\ln[f(t)^2] = -2\ln[f(t)], \tag{34}$$

$$c_3(t) = -\mathrm{i}\int_0^t \frac{1}{M_{\mathrm{eff}}(t')f(t')^2}\,\mathrm{d}u. \tag{35}$$

By the use of the adiabatic approximation and by final averaging over the trigonometric functions we find that the contribution of $c_1(t)$ and $c_3(t)$ to the time evolution of the operator to $\hat{U}(t, 0)$ is negligible and the main contribution follows from $c_2(t)$. We find also that within the adiabatic approximation the effect of change in frequency is quite small and the main effect of squeezing is due to the increase of amplitude.

To show the magnitude of squeezing, it is convenient to use dimensionless coordinates by which (13) can be written as

$$\hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \qquad \hat{p} = -\mathrm{i}\frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}}. \tag{36}$$

Then $\hat{J}_0$ is given by

$$\hat{J}_0 = \frac{\hat{a}^2 - a^{\dagger 2}}{4}, \tag{37}$$

and the unitary operator $\hat{U}(t, 0)$ is given according to (9) and (34) by

$$\hat{U}(t, 0) = \exp\left\{-\tfrac{1}{2}\ln[f(t)](\hat{a}^2 - a^{\dagger 2})\right\}. \tag{38}$$

Equation (38) represents the well-known squeezing operator [31] with a real squeezing parameter $r = -\ln[f(t)]$. Following well-known derivations for the unitary squeezing operator [31] the standard deviations of $\Delta x(t)$ and $\Delta p(t)$ are developed as

$$\Delta x(t) = \frac{1}{\sqrt{2}}\exp[\ln(f(t)] = \frac{1}{\sqrt{2}}f(t), \qquad \Delta p(t) = \frac{1}{\sqrt{2}}\exp[-\ln(f(t)] = \frac{1}{\sqrt{2}f(t)}. \tag{39}$$

Similar results to those of (39) have been given also in previous works [8, 21] in which the relations

$$\Delta x(t) = \left|\exp\left(-\frac{c_2(t)}{2}\right)\right|, \qquad \Delta p(t) = \left|\exp\left(\frac{c_2(t)}{2}\right)\right| \tag{40}$$

have been derived. However the value of $c_2(t)$ was not evaluated in previous works. We find that the squeezing effect, in the adiabatic approximation, is a very strong effect as the standard deviation for $\Delta x(t)$ increases proportional to the increase in the harmonic oscillator amplitude (given in the present analysis by $f(t)$).

## 5. Mechanical oscillator under 'impact' perturbation

In the 'impact' approximation the harmonic oscillator Hamiltonian is exchanged into the 'impact' Hamiltonian given as

$$\hat{H} = \frac{\hat{p}^2}{2M} - F(t)x, \tag{41}$$

where we have neglected here the harmonic potential term $\frac{1}{2}M\omega^2\hat{q}^2$ and $x$ is the location coordinate of the mass $M$. This approximation might be valid for gravitational waves detection by Michelson interferometers.

Gravitational waves are propagating fluctuations of gravitational fields, that is, 'ripples' in spacetime which travel with the speed of light. Everybody in the path of such a wave feels a 'tidal' gravitational force that acts perpendicular to the waves direction of propagation;

these forces change the distance between points, and the size of the changes is proportional to the distance between these points thus gravitational waves can be detected by devices which measure the induced length changes. A promising form of gravitational wave detector uses laser beams to measure the distance between two well-separated masses. Such devices are basicly kilometer sized laser interferometers consisting of three masses placed in L-shaped configuration. The laser beams are reflected back and forth between the mirrors attached to the masses, where the mirrors lying several kilometers away from each other. A gravitational wave passing through this interferometer will cause the length of the arms to oscillate with time. For a polarized gravitational when one arm contracts the other expands and this pattern alternates. The result is that the interference pattern of the two laser beams changes with time. It is expected that laser interferometric detectors are those that will provide us with the first direct detection of gravitational waves on earth.

One way of understanding the effects of gravitational waves operating on interferometers is to describe them as *tidal forces* operating on test masses, i.e. the interferometer mirrors. Typically the gravitational waves frequencies which might be detected by Michelson interferometers on Earth are about some tens of Hz to some KHz, while the pendulum period on which the mirrors are located is about $1/2$ Hz. Under these conditions the effect of gravitational waves on the test masses can be taken into account approximately by the use of the Hamiltonian (41).

In a classical analysis $-F(t)x$ might be considered as the potential energy produced by the gravitational wave. The force in this approximation is given by $-\frac{\mathrm{d}}{\mathrm{d}x}(-F(t)x) = F(t)$ where the force depends only on time. It is quite well known from the field of quantum optics that Hamiltonian of the form $-F(t)x$ does not lead to squeezing effects in electromagnetic waves and such forces are defined as driving forces. One can also verify that the addition of such terms to the mechanical Hamiltonian (1) does not change the squeezing effects [20–22]. It will be, however, of interest to analyze the perturbation induced by the Hamiltonian (41). While an electromagnetic wave packet is propagating in free space without any dispersion, mechanical wave packet is broadened during propagation in free space and we would like to show the magnitude of this effect.

The wavefunction of a mechanical wave packet can be described as

$$\psi = \int a(k) \exp\left\{\exp\left[\mathrm{i}\left(kx - \frac{k^2}{2M}t\right)\right]\right\} \mathrm{d}k. \tag{42}$$

Here $k^2/2M$ is the energy (in units of $\hbar = 1$) and as it is not proportional to $k$ it leads to the dispersion of the wave packet. $a(k)$ describes the distribution of the wave packet in $k$ space and it is fixed by the initial physical conditions. Each component

$$\psi_k = \exp\left[\mathrm{i}\left(kx - \frac{k^2}{2M}t\right)\right] \tag{43}$$

satisfies the Schrödinger equation for free particle as

$$\hat{H}_0\psi_k = \frac{\hat{p}^2}{2M}\psi_k = \frac{k^2}{2M}\psi_k, \qquad \mathrm{i}\frac{\mathrm{d}\psi_k}{\mathrm{d}t} = \frac{k^2}{2M}\psi_k. \tag{44}$$

In the 'impact' approximation we add the term $-F(t)x$ to $H_0$ so that the total Hamiltonian $H$ is given by (41). It is convenient to see the change in the momentum $\hat{k}$ due to the 'impact' perturbation in the Heisenberg picture where

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{k} = [\hat{k}, \hat{H}] = [\hat{k}, -F(t)x] = \mathrm{i}F(t). \tag{45}$$

The momentum $\hat{k}$ is changed therefore in the 'impact' approximation into $\hat{k}'$given by

$$\hat{k}' = \hat{k} + \int_0^t F(t')\,\mathrm{d}t', \tag{46}$$

and this change is independent of $\hat{k}$. By taking expectation values of (46) and dividing by $M$ we get

$$\langle k'/M \rangle = v' = \langle k/M \rangle + \left\langle \int_0^t F(t')\, \mathrm{d}t'/M \right\rangle = v + \left\langle \int_0^t F(t')\, \mathrm{d}t'/M \right\rangle, \qquad (47)$$

where $v$ and $v'$ are the velocities before the 'impact' perturbation, and that obtained by the impact perturbation, respectively. We find that the effect of $-F(t)x$ is to produce additional velocities $\left\langle \int_0^t F(t')\, \mathrm{d}t' \big/ M \right\rangle$ of the wave packets and correspondingly to a movement of the wave packet. One can use (45) for calculating the change in the second moment of $\hat{k}$ and find that it is not changed by the 'impact' perturbation as

$$\frac{\mathrm{d}\langle \hat{k}^2 \rangle}{\mathrm{d}t} = 2\left\langle \hat{k}\frac{\mathrm{d}\hat{k}}{\mathrm{d}t} \right\rangle = 2\langle \hat{k} \rangle F(t) = \frac{\mathrm{d}\langle \hat{k} \rangle^2}{\mathrm{d}t}. \qquad (48)$$

In a similar way one finds that all higher moments of $\hat{k}$ are not changed and the effect of the 'impact' perturbation is to lead to a shift of the wave packet without changing its shape in the momentum $k$ distribution. In the Schrödinger picture the wavefunction in the $x$ representation is changed into

$$\psi = \int a(k) \exp\left\{ \exp\left[ \mathrm{i}\left( k'x - \frac{k'^2}{2M}t \right) \right] \right\} \mathrm{d}k, \qquad (49)$$

where $k' = k + \int_0^t F(t')\, \mathrm{d}t'$.

There is, however, a broadening of the wave packet in the $x$ representation which is like that of a free particle. Such broadening in the $x$ representation is given for an initial Gaussian wave packet by [32]

$$\Delta x = \Delta x_0 \sqrt{1 + \frac{\hbar^2 t^2}{M^2 \Delta x_0^4}}, \qquad (50)$$

where $\Delta x_0$ is the width of the wave packet at the initial time $t = 0$. $t$ is the time in which the wave packet is broadened, $M$ is the mass of the free particle and we have inserted here the proper dimensions including $\hbar$. Let us put some numbers for estimating broadening effects in the detection of gravitational waves by Michelson interferometers [33–35]. Assuming orders of magnitudes: $\Delta x_0 = 10^{-17}$ cm (note this extremely small value which might be expected in the detection of gravitational waves on earth), $t = 10^{-5}$ s, $M = 10^4$ gm, $\hbar = 10^{-27}$ erg s, then we get $\Delta x = \Delta x_0 \sqrt{1 + 10^{-4}}$ where the correction is quite negligible. But if we enlarge the time to $t = 10^{-3}$ s then the correction becomes significant. In conclusion physical experiments with single macroscopic objects became so accurate that quantum-mechanical uncertainty fluctuations should be taken into account.

## 6. Summary, discussion and conclusion

The theory of harmonic oscillators with time-dependent frequency and effective mass has been developed in previous studies [1–28]. Such quantum theories have been used for explaining damping and pumping effects in non-conservative systems. It has been shown also that such systems under certain conditions will produce squeezing phenomena, i.e., the quantum fluctuations in one quadrature will be reduced on the expense of increasing the quantum noise in the other quadrature. The general theory for such effects has been reviewed in the present paper in sections 2 and 3.

In the present work two models for describing the effects of external perturbations on harmonic oscillators are analyzed: (a) in section 4 the use of the adiabatic approximation was developed where the changes in the harmonic oscillator parameters can be neglected during the time period of oscillation but their effect is accumulated over many time periods and leads to squeezing effects. It is shown that the adiabatic approximation can be related to previous studies of time-dependent harmonic oscillator but the use of the adiabatic approximation enable us to simplify the analysis and to get the magnitude of the squeezing effects. This analysis can have implications for mechanical oscillators which are used in traps [30] and for resonant detectors. While the use of squeezed states of radiation in Michelson interferometers has been studied extensively [36–38] possible mechanical squeezing phenomena in resonant detectors [35, 39] were not suggested. (b) In the 'impulse' approximation it is assumed that the oscillators are changing in a time which is short relative to the time period of the harmonic oscillator. In section (5) an Hamiltonian describing impulse driving forces is described which does not lead to any squeezing but broadening effects for the mechanical wave packets might be important, under very special conditions. This model can be related to the detection of gravitational waves in Michelson interferometers.

I hope that the general treatments of mechanical oscillators with perturbations under the adiabatic or impact approximation would be of interest, especially for mechanical oscillators in traps and for resonant detectors.

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## References

[1] Caldirola P 1983 Quantum theory of nonconservative systems *IL Nuovo Cimento* B **77** 241
[2] Kanai E 1948 On the quantization of the dissipative systems *Prog. Theor. Phys.* **3** 440
[3] Hasse R W 1975 On the quantum mechanical treatment of dissipative systems *J. Math. Phys.* **16** 2005
[4] Dodonov V V and Manko V I 1979 Coherent states and the resonance of a quantum damped oscillator *Phys. Rev.* A **20** 550
[5] Remaud B 1980 Damping of the wave packet motion in a general time-dependent quadratic field *J. Phys. A: Math. Gen.* **13** 2013
[6] Abdalla M S 1986 Canonical treatment of harmonic oscillator with variable mass *Phys. Rev.* A **33** 2870
[7] Abdalla M S 1986 Time-dependent harmonic oscillator with variable mass under the action of driving force *Phys. Rev.* A **34** 4598
[8] Cheng C M and Fung P C W 1988 The evolution operator technique in solving the Schrödinger equation, and its application to disentangling exponential operators and solving the problem of a mass-varying harmonic oscillator *J. Phys. A: Math. Gen.* **21** 4115
[9] Baseia B, De Brito A L and Bagnato V S 1992 Squeezing and antisqueezing for a harmonic oscillator having a sudden change of mass *Phys. Rev.* A **45** 5308
[10] Yuen H P 1976 Two-photon coherent states of the radiation field *Phys. Rev.* A **13** 2226
[11] Lewis H R Jr and Reisenfeld W B 1969 An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field *J. Math. Phys.* **10** 1458
[12] Eliezer C J and Gray A 1976 A note on the time-dependent harmonic oscillator *SIAM J. Appl. Math.* **30** 463
[13] Lewis H R Jr 1968 Class of exact invariants for classical and quantum time-dependent harmonic oscillators *J. Math. Phys.* **9** 1976
[14] Lewis H R Jr 1967 Classical and quantum systems with time-dependent harmonic-oscillator-type Hamiltonians *Phys. Rev. Lett.* **18** 510
[15] Yeon K H, Kim H J, Um C I, George T F and Pandey L N 1994 Wavefunction in the invariant representation and squeezed-state function of the time dependent harmonic oscillator *Phys. Rev.* A **50** 1035
[16] Ma X and Rhodes W 1989 Squeezing in harmonic oscillators with time-dependent frequencies *Phys. Rev.* A **39** 1941
[17] Leach P G L 1977 Invariants and wavefunctions for some time-dependent harmonic oscillator-type Hamiltonians *J. Math. Phys.* **18** 1902

[18] Moya-Cessa H and Guasti M F 2003 Coherent states for the time-dependent harmonic oscillator: the step function *Phys. Lett.* A **311** 1
[19] Dattoli G, Solimeno S and Torre A 1986 Algebraic time-ordering techniques and harmonic oscillator with time-dependent frequency *Phys. Rev.* A **34** 2646
[20] Lo C F 1993 Propagator of the general driven time-dependent oscillator *Phys. Rev.* A **47** 115
[21] Lo C F 1991 Generating displaced and squeezed number states by a general driven time-dependent oscillator *Phys. Rev.* A **43** 404
[22] Lo C F 1993 Coherent-state propagator of the generalized time-dependent parametric oscillator *Europhys. Lett.* **24** 319
[23] Abdalla M S and Colegrave R K 1993 Squeezing induced in a harmonic oscillator by a sudden change in mass or frequency *Phys. Rev.* A **48** 1526
[24] Dantas C M A, Pedrosa I A and Baseia B 1992 Harmonic oscillator with time-dependent mass and frequency and a perturbative potential *Phys. Rev.* A **45** 1320
[25] Ji J-Y, Kim J K and Kim S P 1995 Heisenberg-picture approach to the exact motion of a time-dependent harmonic oscillator *Phys. Rev.* A **51** 4268
[26] Sheng D, Khan R D, Jialun Z and Wenda S 1995 Quantum harmonic oscillator with time-dependent mass and frequency *Int. J. Theor. Phys.* **34** 355
[27] Dantas C M A, Pedrosa I A and Baseia M 1992 Harmonic oscillator with time-dependent mass and frequency *Brazilian J. Phys.* **22** 33
[28] Song D-Y 1999 Unitary relations in time dependent harmonic oscillators *J. Phys. A: Math. Gen.* **32** 3449
[29] Saleh B E A and Teich M C 1991 *Fundamentals of Photonics* (New York: Wiley)
[30] Leibfried D, Blatt R, Monroe C and Wineland D 2003 Quantum dynamics of single trapped ions *Rev. Mod. Phys.* **75** 281
[31] Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer)
[32] Bohm D 1955 *Quantum Theory* (New York: Prentice Hall)
[33] Maggiore M 2008 *Gravitational Waves, vol I: Theory and Experiments* (Oxford: Oxford University Press)
[34] Ohanian H C and Ruffini R 1994 *Gravitation and Spacetime* (New York: Norton)
[35] Saulson P R 1994 *Fundamentals of Interferometric Gravitational Wave Detectors* (Singapore: World Scientific)
[36] Assaf O and Ben-Aryeh Y 2002 Quantum mechanical noise in coherent-state and squeezed-state Michelson interferometers *J. Opt. B: Quantum Semiclass. Opt.* **4** 49
[37] Assaf O and Ben-Aryeh Y 2002 Reduction of quantum noise in the Michelson interferometer by the use of squeezed vacuum states *J. Opt. Soc. Am.* B **19** 2716
[38] Barak R and Ben-Aryeh Y 2008 Photon statistics and entanglement in coherent-squeezed linear Mach–Zehnder and Michelson interferometers *J. Opt. Soc. Am.* B **25** 361
[39] Braginski V B and Khalili F Y 1992 *Quantum Measurement* (Cambridge: Cambridge University Press)